REMARKS ON n-DIMENSIONAL FEYNMAN DIAGRAMS, FOR EXAMPLE, WHICH WILL APPEAR IN M-THEORY AND IN F-THEORY

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ABSTRACT. We state some remarks on 'n-dimensional Feynman diagrams' ($n \in \mathbb{N}$).

'n-dimensional Feynman diagrams' ($n \in \mathbb{N}$) will be used in physics in the near future. Here, we let 1-dimensional Feynman diagrams mean Feynman diagrams in 'usual' QFT (see [3][11]). Furthermore, we let 2-dimensional Feynman diagrams mean world sheets in 'usual' superstring theory (see [4][12]). We introduce 'n-dimensional Feynman diagrams' as a generalization of the 1-, 2-dimensional Feynman diagrams as follows.

F-theory, M-theory (see [6] [5] [14] [17] [18] [20]) etc. imply that particles are represented by manifolds whose dimensions are greater than one. Here, consider interaction of these particles and use perturbation theory like the 0-,1-dimensional particle case. Then we will be able to use manifolds whose dimensions are greater than two in order to represent the interaction. In this paper we call these manifolds n-dimensional Feynman diagrams or n-dimensional world membranes if the manifolds are n-dimensional ones. We may use not only manifolds but also CW complexes for particles and Feynman diagrams (See [2] for CW complexes).

Suppose we will complete the high-dimensional particle theory (F-theory, M-theory etc.) without using perturbation theory or Feynman diagrams. However, the limit of the theories is 'usual' superstring theory or 'usual' QFT. Therefore, we will be able to consider n-dimensional Feynman diagrams.

Anyway, mathematically we can discuss n-dimensional Feynman diagrams.

In this paper we state some remarks on 'n-dimensional Feynman diagrams.'

We consider the case where we make two 3-vertex functions into a 4-vertex function. In this case there is a different feature between in the case of high dimensional Feynman diagrams and in the 1-, 2-dimensinal case.

Let m be any integer greater than two. Let M be a compact oriented connected m-manifold with boundary. Let $\partial M = A_1 \amalg A_2 \amalg A_3$, where $\amalg$ denotes a disjoint union and $A_1(resp.A_2, A_3)$ is a connected closed oriented manifold. Let the orientation of A_i be induced from M. Suppose that there is an orientation reversing diffeomorphism $A_i \to A_j$, where we do not assume $i = j$ or $i \neq j$. Let M' be diffeomorphic to M. Let $\partial M' =$

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$A'_1 \amalg A'_2 \amalg A'_3$ and $A_l = A'_l (l = 1, 2, 3)$. Take M and M'. Identify A_i and A'_j by an orientation reversing diffeomorphism $f : A_i \to A'_j$ and obtain a compact oriented connected m-manifold W_f from M and M'. Let $\mathcal{S}_M$ be the set whose elements are the diffeomorphism classes of such W_f

Theorem 1. *For $m \geqq 3$, there is a compact oriented connected m-manifold M such that the above set $\mathcal{S}_M$ is an infinite set.*

Note. Theorem 1 means that, for $m \geqq 3$, two 3-vertx functions can make infinitely many kinds of 4-vertex functions under some conditions. If $m = 2$, such M does not exist as string theorists and topologists know. For $m = 1$, they can say such does not.

Proof. The $m = 3$ case: Take a solid torus. Remove two open 3-balls from the solid torus, call it M. Note $\partial M = S^2 \amalg S^2 \amalg T^2$. Let $f : T^2 \to T^2$ be a diffeomorphism. Let Z be any 3-dimensional Lens space (see [13] for Lens spaces). Take any oriented manifold which is made from Z by removing four open 3-balls, call it Z'. Then $Z' \in \mathcal{S}$. Hence there are countably infinitely many Z', using the homology groups of Z'. Hence Theorem 1 is true in the $m = 3$ case.

The $m > 3$ case: Take $D^2 \times T^{m-2}$, where T^{m-2} is an $(m-2)$-dimensional torus. Remove two open m-balls from $D^2 \times T^{m-2}$, call it M. Then $\mathcal{S}$ includes all manifolds which are made from all [(Lens spaces) $\times T^{m-3}$] by removing four open m-balls, call it Z'. Hence there are countably infinitely many Z', using the homology groups of Z'. Note $\mathcal{S}$ is an infinite set. Hence Theorem 1 is true in the $m > 3$ case. This completes the proof of Theorem 1.

We consider the case where we make 3-vertex functions into an l-vertex function ($l \in \mathbb{N} \cup \{0\}$). Here, let the number of kinds of 3-vertex functions be finite. In this case there is a different feature between in the case of high dimensional Feynman diagrams and in the 1-, 2-dimensinal case, too.

Let m be any integer greater than two. Let $M_i (i = 1, ..., \mu)$ be a compact oriented connected m-dimensional manifold with boundary, where $\mu \in \mathbb{N}$. Let $\partial M_i = \amalg_{j=1}^{j=\nu_i} M_{ij}$, where $\nu_i \in \mathbb{N}$, $\nu_i \geqq 3$, M_{ij} is a connected closed oriented manifold, and the orientation of M_{ij} is induced from that of M_i. Let $B = \amalg B_z (z \in \mathbb{N}, z \geqq 3)$ be a closed oriented $(m-1)$-manifold.

We define a set $\mathcal{W}_{\{M_i\},B}$: An arbitrary element $\in \mathcal{W}_{\{M_i\},B}$ is a compact connected oriented manifold with boundary B with the following properties; There are embedded closed $(m-1)$ manifolds $Y_1, ..., Y_\alpha \subset \mathrm{Int}W$, where $\mathrm{Int}W$ means the interior of W and $Y_i \cap Y_j = \phi$ for $i \neq j$. Let $N(Y_i) = Y_i \times [-1, 1]$ be the neighborhood of Y_i in W. Take $W - \mathrm{Int}N(Y_i) = W_1 \amalg ... \amalg W_w$. Then each W_l is diffeomorphic to M_i for an i.

Let $\mathcal{X}_B$ be a set of all compact oriented connected m-manifolds with boundaries B.

Theorem 2. *Let m, M_i, B, $\mathcal{W}_{\{M_i\},B}$, and $\mathcal{X}_B$ be as above. Then, for any B and any M_i, we have $\mathcal{W}_{\{M_i\},B} \neq \mathcal{X}_B$.*

Note. If $m = 2$, for any B there exists a manifold M such that , $\mathcal{W}_{\{M\},B} = \mathcal{X}_B$, as topologists and string theorists know. Here, $\{M\}$ is a set which has only one element M. Theorem 2 means that we may need a new discussion to divide complex Feynman diagrams into fundamental parts.

Proof. Let $W \in \mathcal{W}_{\{M_i\},B}$. Then we can divide W into pieces W_i, $N(Y_j)$ as above and can regard $W = W_1 \cup ... \cup W_w$. Consider the Meyer-Vietoris exact sequence (see [2][9][16] for the Meyer-Vietoris exact sequence):
$H_j(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \to H_j(\amalg_{i=1}^{i=w} W_i; \mathbb{Q}) \to H_j(W; \mathbb{Q})$. Here, $\amalg_{i,i'}$ means the disjoint unions of $W_i \cap W_{i'}$ for all (i, i'). Consider
$H_1(W; \mathbb{Q}) \to H_0(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \to H_0(\amalg W_i; \mathbb{Q}) \to H_0(W; \mathbb{Q}) \to 0$. Note
$H_0(\amalg W_i; \mathbb{Q}) \cong \mathbb{Q}^w$ and $H_0(W; \mathbb{Q}) \cong \mathbb{Q}$.

Suppose that $\partial W = B$ has z components as above, that is, it is a Feynman diagrams with z outlines. Hence $H_0(\amalg_{i,i'}\{W_i \cap W_{i'}\}; \mathbb{Q}) \cong \mathbb{Q}^\rho$, where $\rho \geqq \frac{3w-z}{2}$.

We suppose Theorem 2 is not true and induce the contradictory. If Theorem 2 is not true, then $\mathcal{W}_{\{M_i\},B}$ includes any compact oriented connected m-manifold with boundary B. If $H_1(W; \mathbb{Q}) \cong \mathbb{Q}^l$, we have the exact sequence:
$\mathbb{Q}^l \to \mathbb{Q}^\rho \to \mathbb{Q}^w \to \mathbb{Q} \to 0$. Hence $l \geqq \rho - w + 1$. Hence $l \geqq \frac{w-z+2}{2}$. Hence $(2l + z - 2) \geqq w$.

Let X be a compact manifold. Take a handle decomposition of X. Consider the numbers of handles in the handle decompositions. Let $h(X)$ be the minimum of such the numbers. Suppose that M is one of the manifolds M_i and that $h(M) \geqq h(M_i)$ for any i. Then we have $w \times h(M) \geqq h(W)$. Hence $(2l + z - 2) \times h(M) \geqq h(W)$.

For any natural number N, there are countably infinitely many compact oriented connected m-manifolds W with boundaries such that ∂W has z components, that
$H_1(W; \mathbb{Q}) \cong \mathbb{Q}^l$, and that $h(W) \geqq N$. Because: There is an n-dimensional manifold P such that $H_1(P; \mathbb{Q}) \cong \mathbb{Q}^l$. There is an n-dimensional rational homology sphere Q which is not an integral homology sphere. Make a connected sum which is made from one copy of P and q copies of Q ($q \in \mathbb{N} \cup \{0\}$).

This is the contradiction. This completes the proof.

We consider the case where we make 3-vertex functions into an l-vertex function ($l \in \mathbb{N} \cup \{0\}$). Here, let the number of kinds of 3-vertex functions be infinite.

Let n be any integer greater than two. Then there is an infinite set $\mathcal{Q}$ which is a proper subset of the set of all n-manifolds (i.e. which is not the set of all n-manifolds) with the following properties: Using elements of a finite subset of $\mathcal{Q}$ in the similar manner to make W from $\{M_i\}$ above Theorem 2, we can construct any compact oriented n-manifold with boundary. Furthermore, we can suppose the boundary of any element of $\mathcal{Q}$ has one, two or three components. We state theorem in the $n = 3$ case. In the $n > 3$ case we have similar theorem. The proof is easy, considering properties of handle decompositions. See [15] [8] for handle decompositions.

Theorem 3. *Let Q_g be a compact oriented 3-manifold whose handle decomposition is $(F_g \times [0,1]) \cup$ (a 1-handle), where F_g denotes a closed oriented surface with genus g. Let $Q_{g,h}$ be a compact connected oriented 3-manifold whose handle decomposition is $(F_g \times [0,1]) \cup (F_h \times [0,1]) \cup$ (a 1-handle).@ Note $\partial Q_g = F_g \amalg F_{g+1}$ and $\partial Q_{g,h} = F_g \amalg F_h \amalg F_{g+h}$. Take a set $\mathcal{Q} = \{B^3, Q_{g,h}, Q_g | g, h \in \mathbb{N} \cup \{0\}\}$, where B^3 is a 3-ball. Let M be an arbitrary compact oriented 3-manifold with boundary. Then M is made from elements of a finite subset of $\mathcal{Q}$ in the similar manner to make W from $\{M_i\}$ above Theorem 2.*

One way to use n-dimensional Feynman diagrams ($n \geqq 3$) is to restrict what kind of compact oriented n-manifolds to represent Feynman diagrams. Indeed, in the $n = 1$ case, we restrict what kind of 'CW-complexes made from 0-cells and 1-cells' represent the Feynman diagrams. See [2] for CW complexes.

We might note the following: Suppose that we use only elements of $\mathcal{R} = \{B^3, Q_{g,}, Q_{g,h}| g, h \in A\}$, where A is a finite subset of $\mathbb{N} \cup \{0\}$. Then any element of $\mathcal{R}$ includes a submanifold which is diffeomorphic to Q_k (resp. $Q_{k,l}$) for any $k, l \in \mathbb{N} \cup \{0\}$. It might not be a good idea to restrict what kind of compact oriented 3-manifolds to represent Feynman diagrams.

Although it is one way of saying, Witten's Chern-Simons theory (see [19]) on 3-manifolds M with the gauge group G are regarded as the theory $M \to \mathcal{G}$, where $\mathcal{G}$ is the Lie ring of G. Recall that $\mathcal{G}$ is a vector space. Note that, in this case, we can regard all compact oriented 3-manifolds with boundaries as 3-dimensional world membranes. It might not be a good idea to restrict unnaturally what kind of compact oriented 3-manifolds to represent Feynman diagrams. For example, we might need an idea that such restriction make a sense in only low-energy case.

In the two dimensional case (i.e. 'usual' string theory) particles are represented both by closed manifolds (i.e. closed strings) and by compact manifolds with boundaries (i.e. open strings). In the n-dimensional case ($n \geqq 3$) particles will be represented both by closed manifolds and by compact manifolds with boundaries. In this paper we concentrate on the case of closed manifolds.

It might be good to suppose that n-dimensional Feynman diagrams are complex manifolds, symplectic ones, Kähler ones, toric ones, hyperbolic ones, or something. However, in these cases, there exist underlying smooth manifolds (and underlying topological manifolds). Hence our theorems in this paper are fundamental restrictions to such the theories, as Pauli exculsion rule and Coleman-Mandula NO-GO theorem are. Because in our theorems n-dimensional Feynman diagrams are just smooth manifolds.

Research on n-dimensional Feynman diagrams in a time-space is connected with that on submanifolds in a manifold. Submanifold theory includes n-dimensional knot theory as an important field. See [1][7][10].

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